

Mass generation by a Lorentz-invariant gas of spacetime defects

F.R. Klinkhamer*

*Institute for Theoretical Physics,
Karlsruhe Institute of Technology (KIT),
76128 Karlsruhe, Germany*

J.M. Queiruga†

*Institute for Theoretical Physics,
Karlsruhe Institute of Technology (KIT),
76128 Karlsruhe, Germany
and*

*Institute for Nuclear Physics,
Karlsruhe Institute of Technology,
Hermann-von-Helmholtz-Platz 1,
76344 Eggenstein-Leopoldshafen, Germany*

Abstract

We present a simple model of defects embedded in flat spacetime, which is designed to maintain Lorentz invariance over large length scales. Still, there are effects from these spacetime defects on the propagation of physical fields, notably mass generation for scalars and Dirac fermions.

* frans.klinkhamer@kit.edu

† jose.queiruga@kit.edu

I. INTRODUCTION

It can be argued that spacetime over small length scales has a nontrivial structure [1–5]. What the precise nature of this small-scale “structure” would be is, however, unclear.

It is known, for example, that static Swiss-cheese-type models affect particle propagation and experimental data strongly constrain the “holes” of such a classical spacetime [6, 7]. As the static holes of such a model violate Lorentz invariance, the above-mentioned bounds strongly constrain Lorentz violation; see also the related discussion in Ref. [8]. The conclusion appears to be that, if spacetime somehow has a small-scale structure, the relevant (quantum) theory manages to keep Lorentz invariance to high precision.

For this reason, it may be of interest to investigate toy models of spacetime-defects, where the models are designed to maintain Lorentz invariance. One class of such models involves pointlike defects, as studied in Refs. [9–11] (see also Refs. [12, 13] for a general discussion). In the present article, we present one further toy model with pointlike defects and study the induced modifications of the standard particle propagation (“standard” referring to the perfect Minkowski spacetime without defects).

II. POISSON DISTRIBUTION AND LORENTZ INVARIANCE

It is a nontrivial issue to find distributions of defects over four-dimensional Minkowski spacetime, which preserve the Lorentz symmetry in the large. Let us assume, for example, that the defects are distributed over a regular lattice in one particular reference frame. Averaged over large scales, the distribution is homogeneous. But if we go to a Lorentz-boosted frame, the density of defects will increase in the direction of the boost, while remaining constant in the perpendicular direction. Apparently, the Lorentz symmetry is broken by having a preferred reference frame in the original setup with a regular lattice.

Still, if the defects are distributed according to a Poisson process (a “sprinkling” procedure), boosts do not break Lorentz invariance [14–16]. The probability of finding n defects in a four-dimensional volume V_4 is then given by

$$P_n(V_4) = \frac{(\rho_d V_4)^n \exp(-\rho_d V_4)}{n!}. \quad (2.1)$$

The parameter ρ_d characterizes the distribution and corresponds to the spacetime density of defects. Note that the Poisson process, for constant parameter ρ_d , depends only on the four-dimensional volume of the region considered. This implies that the probability of finding n defects contained in a region of volume V_4 is invariant under volume-preserving transformations. Since Lorentz transformations preserve the spacetime volume, the sprinkling is Lorentz invariant. Phrased in a different way, the defect distribution from the Poisson process has no built-in “structure.” If present, such a built-in structure would be deformed by Lorentz contraction, just as for the lattice setup discussed above.

We see immediately from the Poisson distribution (2.1) that, on average, the typical number of defects inside a region of volume V_4 is given by $\langle n \rangle_{V_4} = \rho_d V_4$ and that the fluctuations of this number are of order $\sqrt{\rho_d V_4}$, so that the relative fluctuations become irrelevant for large V_4 .

III. EFFECTIVE MODEL FOR A GAS OF SPACETIME DEFECTS

A. General remarks

The explicit calculation of physical observables in a theory with finite-size spacetime defects is prohibitively difficult. We can, instead, use a simple model with a gas of point-like defects [9, 11].

In the model discussed here, the spacetime defects are represented by delta functions in a classical background Minkowski spacetime, where the delta functions are coupled to a “mediator” real scalar field $\sigma(x)$ with random charges $\epsilon_n \in \{-1, 1\}$ and a coupling constant λ . The charges of individual defects are randomly chosen, so that the total charge vanishes over a large enough spacetime volume. The mediator field $\sigma(x)$ is also coupled to three “physical” fields: a massless real scalar field $\phi(x)$ with a nonderivative quartic-coupling term, a massless Dirac fermion field $\psi_1(x)$ with a Yukawa term, and a massless Dirac fermion field $\psi_2(x)$ with a nonrenormalizable Yukawa-type term.

In short, a nontrivial spacetime with defects is modelled by a perfect classical Minkowski spacetime and an action with delta functions coupled to a real scalar field $\sigma(x)$. In turn, this mediator field $\sigma(x)$ is coupled to physical fields $\phi(x)$ and $\psi_a(x)$, all fields propagating over Minkowski spacetime.

B. Massless mediator field

The following effective action is considered:

$$S_{\text{eff}} = \int_{\mathbb{R}^4} d^4x \left(\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + i \bar{\psi}_1 \gamma^\mu \partial_\mu \psi_1 + i \bar{\psi}_2 \gamma^\mu \partial_\mu \psi_2 \right. \\ \left. + \frac{1}{2} \eta^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma + \lambda \sigma \left[\sum_{n=1}^{\infty} \epsilon_n \delta^{(4)}(x - x_n) \right]_{\{x_n\} \text{ from Poisson}} \right. \\ \left. + g_s \sigma^2 \phi^2 + g_{f,1} \sigma \bar{\psi}_1 \psi_1 + g_{f,2} \lambda \sigma^2 \bar{\psi}_2 \psi_2 \right), \quad (3.1a)$$

$$\epsilon_n \in \{-1, 1\}, \quad (3.1b)$$

$$\sum_{n=1}^{\infty} \epsilon_n = 0, \quad (3.1c)$$

with the Minkowski metric $\eta_{\mu\nu} \equiv [\text{diag}(1, -1, -1, -1)]_{\mu\nu}$ for standard Cartesian coordinates x^μ . Throughout, we use natural units with $\hbar = 1 = c$.

In the action (3.1a), the cores of the spacetime defects are modeled by Dirac delta functions centered at the points x_1, x_2, \dots of Minkowski spacetime. As discussed in Sec. II, these points are distributed according to a Poisson process (sprinkling), in order to preserve the Lorentz symmetry. The long-range effects of the defect cores are modelled by a real scalar field σ with random charges $\epsilon_n \in \{-1, 1\}$ and coupling strength λ . The scalar field σ mediates between the defect cores and the physical fields ϕ and ψ_a .

The massless scalar fields ϕ and σ in (3.1a) have mass dimension 1, the massless fermionic fields ψ_a have mass dimension $3/2$, the coupling constant λ has mass dimension -1 and the couplings g_s and $g_{f,a}$ are dimensionless. As mentioned in Sec. III A, the idea behind the action (3.1a) is that a nontrivial spacetime (manifold or not) is modelled by delta functions located at the spacetime points x_n of the Minkowski manifold and by a mediator field $\sigma(x)$. The mediator field $\sigma(x)$ is coupled to the delta functions and to additional physical fields $\phi(x)$ and $\psi_a(x)$, with all fields propagating over classical Minkowski spacetime and the interaction terms given by the last three terms of (3.1a).

In order to recover the standard perturbative results by use of Feynman diagrams, the interactions terms of (3.1a) essentially need to be “turned off” in the asymptotic regions [17]. This can be done by making the couplings in (3.1a) spacetime dependent,

$$\{\lambda(x), g_s(x), g_{f,a}(x)\} = \begin{cases} \{\bar{\lambda}, \bar{g}_s, \bar{g}_{f,a}\} & \text{for } x \in V_{4,\text{cutoff}}, \\ \{0, 0, 0\} & \text{otherwise,} \end{cases} \quad (3.2)$$

where the barred quantities on the right-hand side are truly constant. The spacetime volume $V_{4,\text{cutoff}}$ in (3.2) is taken to be suitably large and the behavior in the transition region can be adequately smoothed. In the following, we will keep this spacetime dependence of the couplings $\{\lambda, g_s, g_{f,a}\}$ implicit. Incidentally, restricting the sums in (3.1) to the finite volume $V_{4,\text{cutoff}}$ makes these sums well-behaved.

The classical solution for the mediator field σ can be easily obtained for $g_s = g_{f,a} = 0$,

$$\frac{\delta S_{\text{eff}}}{\delta \sigma} = 0 \Rightarrow \square \sigma = \lambda \sum_n \epsilon_n \delta^{(4)}(x - x_n). \quad (3.3)$$

The solution is given by

$$\sigma(x) = \sigma_0(x) + \lambda \sum_m \epsilon_m \int d^4 x' G(x, x') \delta^{(4)}(x - x_m), \quad (3.4)$$

where $\sigma_0(x)$ is the free solution [corresponding to the homogeneous equation (3.3) with $\lambda = 0$] and $G(x, x')$ is a Green’s function for the d’Alembert operator \square ,

$$G(x, x') = \frac{1}{4\pi^2 |x - x'|^2}. \quad (3.5)$$

Finally, the solution of (3.3) takes the following form:

$$\sigma(x) = \sigma_0(x) + \lambda \sum_n \frac{\epsilon_n}{4\pi^2 |x - x_n|^2} \equiv \sigma_0(x) + \sigma_c(x). \quad (3.6)$$

Let us now determine the two-point function for σ . In the quantization procedure, the free part of σ can be split in positive and negative frequency modes as usual,

$$\sigma_0(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p e^{-ipx} + a_p^\dagger e^{ipx}) . \quad (3.7)$$

If we define

$$j(x) \equiv \lambda \sum_n \epsilon_n \delta^{(4)}(x - x_n) , \quad (3.8)$$

the Fourier transform takes a simple form

$$j(p) = \int d^4x e^{ipx} j(x) = \lambda \sum_n \epsilon_n e^{ipx_n} . \quad (3.9)$$

With the help of (3.9), we can expand the correction term in (3.6) as follows:

$$\sigma_c(x) = \int \frac{d^3p}{(2\pi)^3} \left(\frac{\lambda \sum_n \epsilon_n e^{ipx_n} e^{-ipx}}{2\omega_p} \mathbb{I} + \text{H.c.} \right) , \quad (3.10)$$

where \mathbb{I} is the identity operator. The only nonvanishing contributions to the two-point function come from terms proportional to $\langle 0 | a_p a_q^\dagger | 0 \rangle = (2\pi^3) \delta^{(3)}(p - q)$ or proportional to $\langle 0 | \mathbb{I}^2 | 0 \rangle = 1$. The final result is given by

$$\begin{aligned} \langle 0 | \sigma(x) \sigma(y) | 0 \rangle &= \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip(x-y)}}{2\omega_p} \\ &\quad + 4\lambda^2 \sum_{m,n} \epsilon_n \epsilon_m \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip(x-x_n)}}{2\omega_p} \int \frac{d^3q}{(2\pi)^3} \frac{e^{-iq(x_m-y)}}{2\omega_q} . \end{aligned} \quad (3.11)$$

Taking into account that the first term in (3.11) corresponds to the free two-point function [denoted by $\Delta(x - y)$ as usual] we get

$$\langle 0 | \sigma(x) \sigma(y) | 0 \rangle = \Delta(x - y) + 4\lambda^2 \sum_{m,n} \epsilon_n \epsilon_m \Delta(x - x_n) \Delta(x_m - y) . \quad (3.12)$$

The expression (3.12) has a simple interpretation: the amplitude for a scalar field to propagate from a spacetime point x to a spacetime point y is given by the free amplitude plus all possible products of the free amplitude of particle propagation from x to the position of a defect x_n times the free amplitude of particle propagation from another defect x_m to y . Under the random phase assumption (see Appendix A), all cross terms joining different defects in (3.12) average to zero. The expression (3.12) then simplifies to

$$\langle 0 | \sigma(x) \sigma(y) | 0 \rangle = \Delta(x - y) + 4\lambda^2 \sum_n \Delta(x - x_n) \Delta(x_n - y) . \quad (3.13)$$

This full tree-level propagator contains the free propagator and the sum of all possible insertions of a single defect.

C. Massive mediator field

For completeness, we also consider the case where the mediator field has a nonzero initial mass m_0 . The massive version of the original action (3.1a) takes the following form:

$$S_{\text{eff}, m_0} = \int_{\mathbb{R}^4} d^4x \left(\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + i \bar{\psi}_1 \gamma^\mu \partial_\mu \psi_1 + i \bar{\psi}_2 \gamma^\mu \partial_\mu \psi_2 + \frac{1}{2} \eta^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma + \frac{1}{2} m_0^2 \sigma^2 \right. \\ \left. + \lambda \sigma \sum_{n=1}^{\infty} \epsilon_n \delta^{(4)}(x - x_n) + g_s \sigma^2 \phi^2 + g_{f,1} \sigma \bar{\psi}_1 \psi_1 + g_{f,2} \lambda \sigma^2 \bar{\psi}_2 \psi_2 \right), \quad (3.14)$$

with the physical fields ϕ and ψ_a still being massless, as long as interactions are neglected.

As before, the classical equation for σ can be written as

$$(\square - m_0^2) \sigma = \lambda \sum_m \epsilon_n \delta^{(4)}(x - x_n). \quad (3.15)$$

The complete solution of the equation can be split in two parts,

$$\sigma(x) = \sigma_0(x) + \lambda \sum_m \epsilon_n \int d^4x' G(x, x') \delta^{(4)}(x - x_n), \quad (3.16)$$

where, now, $\sigma_0(x)$ is the solution of the homogeneous equation $(\square - m_0^2) \sigma_0(x) = 0$ and $G(x, x')$ is a Green's function of the operator $\square - m_0^2$,

$$G(x, x') = \frac{m_0 \mathcal{K}_1(m_0 |x - x'|)}{4\pi^2 |x - x'|}, \quad (3.17)$$

with \mathcal{K}_1 the modified Bessel function of the second kind. A calculation similar to the one of Sec. IIIB gives the following result for the two-point function:

$$\langle 0 | \sigma(x) \sigma(y) | 0 \rangle_{m_0} = \Delta_{m_0}(x - y) + 4\lambda^2 \sum_{n,m} \epsilon_n \epsilon_m \Delta_{m_0}(x - x_n) \Delta_{m_0}(x_m - y), \quad (3.18)$$

where $\Delta_{m_0}(x - y)$ is the free massive propagator. Under the random phase assumption (see Appendix A), the expression (3.18) simplifies to

$$\langle 0 | \sigma(x) \sigma(y) | 0 \rangle_{m_0} = \Delta_{m_0}(x - y) + 4\lambda^2 \sum_n \Delta_{m_0}(x - x_n) \Delta_{m_0}(x_n - y), \quad (3.19)$$

which has the same structure as expression (3.13) for the massless-mediator case.

IV. MASS GENERATION FOR THE MEDIATOR FIELD

As seen in Sec. III, the interaction of the mediator field $\sigma(x)$ with the delta functions leads to a nontrivial modification of the σ propagator. Let us focus on the massless case given by the action (3.1a). The propagator for the mediator field (3.13) can then be rewritten as follows:

$$\Delta(x, y) = \Delta_0(x - y) + 4\lambda^2 \Delta_1(x, y), \quad (4.1)$$

where $\Delta_0(x-y)$ corresponds to the free propagator and $\Delta_1(x, y)$ to the correction from the interactions with the defects. We will see that this last term generates a nonzero mass for the σ field.

If a mass m_σ is indeed generated, then the following equation must hold:

$$(\square_x - m_\sigma^2) \Delta(x, y) = -\delta^{(4)}(x - y), \quad (4.2)$$

for $m_\sigma^2 \neq 0$. After inserting the propagator (4.1) in (4.2), we obtain to order λ^2

$$4\lambda^2 \sum_n \delta^{(4)}(x - x_n) \Delta_0(x_n - y) - m_\sigma^2 \Delta_0(x - y) + \mathcal{O}[\lambda^4 \rho_d^2 \Delta_1(x, y)] = 0. \quad (4.3)$$

It is still not easy to interpret the first term on the left-hand side of (4.3). To do so, we can use the fact that the points x_n are distributed according to a Poisson process as discussed in Sec. II. According to (2.1), the number of defects grows with the spacetime volume, $dn \propto d^4x$, where the proportionality factor is given by the density parameter ρ_d . Furthermore, the distribution is assumed to be dense and, therefore, the characteristic distance between defects, $l_d \equiv \rho_d^{-1/4}$, is assumed to be small. This allows us to approximate the sum in (4.3) by an integral,

$$\sum_n \rightarrow \rho_d \int_{\mathbb{R}^4} d^4x. \quad (4.4)$$

Applying (4.4) to (4.3) we obtain the result

$$m_\sigma^2 = 4\lambda^2 \rho_d. \quad (4.5)$$

The first corrections to the mass-square (4.5) will appear as loop corrections involving the dimensionless couplings g_s and $g_{f,a}$.

We conclude that, as a result of the interactions with the delta functions, the mediator field σ has acquired a mass. This result can be confirmed by working directly in the momentum-space representation. Taking (4.4) into account, we get the following expression for the full propagator:

$$\langle 0 | \sigma(x) \sigma(y) | 0 \rangle = \Delta(x - y) + 4\lambda^2 \rho_d \int d^4z \Delta(x - z) \Delta(z - y). \quad (4.6)$$

After shifting the z variable (in order to make explicit the dependence of the two-point function on the difference $x - y$), we can rewrite (4.6) in momentum space as follows:

$$G(p) = \frac{1}{p^2 + i\epsilon} + 4\lambda^2 \rho_d \int d^4z \Delta(z) \frac{e^{ipz}}{p^2 + i\epsilon}. \quad (4.7)$$

Finally, integration with respect to z gives

$$G(p) = \frac{1}{p^2 + i\epsilon} - \frac{1}{p^2 + i\epsilon} 4\lambda^2 \rho_d \frac{1}{p^2 + i\epsilon}. \quad (4.8)$$

$$\Sigma_\phi(p) = \text{[Diagram 1]} + \text{[Diagram 2]}$$

Figure 1. Self-energy contributions for the physical scalar ϕ . The dashed propagator corresponds to the mediator scalar field σ and the counterterm is given in Fig. 2.

$$\overline{p} \text{---} \bigotimes \text{---} p = p^2 \delta_\phi - \delta_{m_\phi}$$

Figure 2. Counterterm for Fig. 1.

The expression (4.8) can be rewritten to quadratic order in λ as follows:

$$G(p) = \frac{1}{p^2 - 4\lambda^2 \rho_d + i\epsilon} + \mathcal{O}[\lambda^4 \rho_d^2 (p^2 + i\epsilon)^{-3}] . \quad (4.9)$$

The mass-square term (4.5) for σ appears in this representation as a pole in the momentum-space propagator. In the limit $\lambda \rightarrow 0$ (no long-range effects of the defect cores) and/or in the limit $\rho_d \rightarrow 0$ (vanishing density of defect cores), the mediator field remains massless.

For the massive case (3.14), the pole in the σ -propagator is already shifted by the initial mass m_0 and an effective mass-square $m_\sigma^2 = m_0^2 + 4\lambda^2\rho_d$ is obtained.

V. PHYSICAL FIELDS

A. Setup

The massless physical fields ϕ and ψ_a only “feel” the presence of the defect cores by interaction with the mediator field σ . From now on, we focus on the effective action corresponding to (3.1a) with vanishing initial mass for σ . We can write this effective action at order λ^2 as follows

$$S_{\text{eff}, \lambda^2} = \int_{\mathbb{R}^4} d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + i \bar{\psi}_1 \gamma^\mu \partial_\mu \psi_1 + i \bar{\psi}_2 \gamma^\mu \partial_\mu \psi_2 + \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma + 2\lambda^2 \rho_d \sigma^2 \right. \\ \left. + g_s \sigma^2 \phi^2 + g_{f,1} \sigma \bar{\psi}_1 \psi_1 + g_{f,2} \lambda \sigma^2 \bar{\psi}_2 \psi_2 \right) + \mathcal{O} [\lambda^4 \rho_d^2] \ , \quad (5.1)$$

where the delta functions have produced a mass term for the σ field as derived in Sec. IV.

B. Physical scalar field ϕ

We now ask what happens to the massless physical field ϕ by its interaction with the mediator field σ . The self-energy for the scalar field is given by Fig. 1. With appropriate reg-

ularization, the last term in Fig. 1 corresponds the counterterm of Fig. 2. This counterterm cancels the divergence coming from the one-loop integral.

Consider Pauli–Villars (PV) regularization. For the 1-loop diagram of Fig. 1, the divergent integral is

$$I(m_\sigma^2) \equiv g_s \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^2 + m_\sigma^2}, \quad (5.2)$$

where k_E is the Euclidean momentum. We now define the PV-regularized integral by

$$I_{\text{PV}}(m_\sigma^2) = I(m_\sigma^2) - I_{\text{PV}}(\Lambda^2) - (m_\sigma^2 - \Lambda^2) I'(\Lambda^2), \quad (5.3)$$

with the Pauli–Villars regulator Λ . Evaluating (5.3) gives

$$I_{\text{PV}}(m_\sigma^2) = -g_s \frac{m_\sigma^2}{(4\pi)^2} + g_s \frac{\Lambda^2}{(4\pi)^2} \left(1 - \log \frac{\Lambda^2}{m_\sigma^2} \right). \quad (5.4)$$

The mass counterterm of Fig. 2 cancels exactly the Λ terms of (5.4),

$$\delta_\phi = 0 \quad \text{and} \quad \delta_{m_\phi} = g_s \frac{\Lambda^2}{(4\pi)^2} \left(1 - \log \frac{\Lambda^2}{m_\sigma^2} \right). \quad (5.5)$$

As a result, a nonzero mass for ϕ is generated at one-loop level,

$$M_\phi^2 \Big|^{(\text{PV reg.})} = \lim_{p \rightarrow 0} \Sigma_\phi(p) \Big|^{(\text{PV reg.})} = g_s \frac{m_\sigma^2}{(4\pi)^2}, \quad (5.6)$$

with $m_\sigma^2 = 4\lambda^2 \rho_d$ from (4.5). Note that the point-splitting regularization and the dimensional regularization give a similar result for the generated scalar mass, with the same parametric dependence $g_s m_\sigma^2$.

The generated mass for the scalar field as given by (5.6) depends linearly on both ρ_d and λ^2 . This implies that, in order to give a nonzero mass to the scalar field, the presence of defect cores ($\rho_d \neq 0$) and the interaction of defect cores with the mediator field ($\lambda \neq 0$) are essential.

Two general remarks are in order. First, the underlying nontrivial spacetime produces not only the model (3.1a) but also the required counterterms such as (5.5). If a single energy scale E_{foam} (equal or not equal to the Planck energy $E_P \equiv G^{-1/2}$) sets the parameters $\lambda \sim 1/E_{\text{foam}}$ and $\rho_d \sim (E_{\text{foam}})^4$, then it is also to be expected that $\Lambda \sim E_{\text{foam}}$ (and the Pauli–Villars “regularization” is no longer a mere mathematical device but is rooted in physical reality). The generated mass-square (5.6) can be very much smaller than $(E_{\text{foam}})^2$ if $g_s \ll 1$.

Second, the following simple question arises: is it not possible that the generated mass-square (5.6) gets absorbed in the square of the renormalized mass and that, thereby, the effects from our spacetime defects become invisible? In general, this is certainly possible, but not for the setup of the theory as outlined in Sec. III. Specifically, the coupling λ is taken to vanish far out, according to (3.2). This means that the renormalized mass relevant to the infinite-volume perfect Minkowski spacetime is unaffected by spacetime defects, as their effects ($\bar{\lambda} \neq 0$) are confined to the finite volume $V_{4, \text{cutoff}}$.

$$\Sigma_{\psi_1}(p) = \text{diagram 1} + \text{diagram 2}$$

Figure 3. Self-energy contributions for the physical fermion ψ_1 . The dashed propagator corresponds to the mediator scalar field σ and the counterterm is given in Fig. 4.

$$\text{diagram} = \not{p} \delta_{\psi_1} - \delta_{m_{\psi_1}}$$

Figure 4. Counterterm for Fig. 3.

C. Physical fermion field ψ_1

The interaction of the massless fermionic field ψ_1 with σ gives rise to different effects compared to those of the scalar field ϕ . The self-energy for the fermion including the counterterm is given by Figs. 3 and 4.

Consider again Pauli–Villars regularization. The unsubtracted regularized integral from the 1-loop diagram of Fig. 3 gives

$$\Sigma_{\psi_1}(p) \Big|^{(\text{PV reg., unsubtr.})} = -\frac{g_{f,1}^2}{16\pi^2} \not{p} \log \left(\frac{\Lambda^2}{m_\sigma^2} \right) - \frac{g_{f,1}^2}{8\pi^2} \not{p} \int_0^1 dz z \log \left(\frac{m_\sigma^2}{m_\sigma^2 - p^2(1-z)} \right), \quad (5.7)$$

where Λ is the Pauli–Villars regulator. The required counterterm is then

$$\delta_{\psi_1} = \frac{g_{f,1}^2}{16\pi^2} \not{p} \log \left(\frac{\Lambda^2}{m_\sigma^2} \right) \quad \text{and} \quad \delta_{m_{\psi_1}} = 0. \quad (5.8)$$

This results in

$$M_{\psi_1} \Big|^{(\text{PV reg.})} = \lim_{p \rightarrow 0} \Sigma_{\psi_1}(p) \Big|^{(\text{PV reg.})} = 0. \quad (5.9)$$

Note that the point-splitting regularization and the dimensional regularization also give a vanishing generated mass for ψ_1 .

Hence, the fermion ψ_1 does not get a mass due to the interaction with the mediator field σ , at least at the 1-loop level and under the assumption that σ does not acquire a vacuum expectation value. Expanding on this last point, the effective action (5.1) is invariant under the following axial transformation:

$$\psi_1(x) \rightarrow \exp[i(\pi/2)\gamma_5] \psi_1(x), \quad (5.10a)$$

$$\sigma(x) \rightarrow -\sigma(x). \quad (5.10b)$$

[Observe that, under this transformation, the linear σ term in the original action (3.1a) is effectively unchanged for an equal number of positive and negative random charges ϵ_n .] If unbroken, the axial symmetry (5.10) of the effective action (5.1) rules out a direct (or generated) mass term $M\bar{\psi}_1\psi_1$.

$$\Sigma_{\psi_2}(p) = \text{fermion line with loop} + \text{fermion line with counterterm}$$

Figure 5. Self-energy contributions for the physical fermion ψ_2 . The dashed propagator corresponds to the mediator scalar field σ and the counterterm is given in Fig. 6.

$$p \text{ --- } \otimes \text{ --- } p = \not{p} \delta\psi_2 - \delta m_{\psi_2}$$

Figure 6. Counterterm for Fig. 5.

D. Physical fermion field ψ_2

The interaction of the massless fermionic field ψ_2 with σ differs from that of ψ_1 with σ , being, in fact, similar to the interaction of the scalar field ϕ with σ .

The self-energy for the fermion including the counterterm is given by Figs. 5 and 6. It is now clear that the ψ_2 self-energy of Fig. 5 has the same structure as the ϕ self-energy of Fig. 1. In other words, the divergent integral for the ψ_2 self-energy is proportional to (5.2).

Consider again Pauli–Villars regularization and take over the relevant results from Sec. VB. Adapting the constants in (5.6), the following result is obtained:

$$M_{\psi_2} \Big|^{(\text{PV reg.})} = \lim_{p \rightarrow 0} \Sigma_{\psi_2}(p) \Big|^{(\text{PV reg.})} = g_{f,2} \lambda \frac{m_\sigma^2}{(4\pi)^2}, \quad (5.11)$$

with $m_\sigma^2 = 4\lambda^2 \rho_d$ from (4.5). The point-splitting regularization and the dimensional regularization give a similar result for the generated fermion mass, with the same parametric dependence $g_{f,2} \lambda m_\sigma^2$.

As the ψ_2 interaction term of the effective action (5.1) involves a factor σ^2 [instead of the single factor σ of the ψ_1 interaction term], the axial transformation (5.10), with ψ_1 replaced by ψ_2 , no longer leaves the action (5.1) invariant. Hence, there is no axial symmetry for the ψ_2 field to exclude the appearance of a ψ_2 mass term.

VI. DISCUSSION

It has become clear over the last years that, if a “quantum spacetime foam” somehow results in a effective classical spacetime manifold with small-scale structure, this effective manifold must be Lorentz-invariant to high precision (at the 10^{-15} level in the photon sector [7]). In the present article, we have, therefore, investigated a model of spacetime defects which is Lorentz-invariant over large enough spacetime volumes. Even though there is no apparent Lorentz violation in this model, there may still be nontrivial effects for the propagation of particles. The quantities that feel the effects of the small-scale structure must

be themselves Lorentz-invariant, an obvious example being mass. Indeed, we have found a generated mass for both a scalar field and a Dirac fermion field (as long as there is no effective axial symmetry of the toy model considered). It is also possible to keep the Dirac fermion field massless, if an effective axial symmetry is built into the toy model.

Incidentally, the mass generation found here is not entirely surprising, as mass generation is known to occur for non-Minkowskian manifolds [18]. The manifold considered in Ref. [18] has nontrivial topology at large length scales (manifold $\mathbb{R}^3 \times S^1$), whereas we are interested in nontrivial topology at small length scales (cf. the discussion in Ref. [9]).

Assuming our results to apply to the Higgs scalar boson of the Standard Model of elementary particle physics (with a mass around 125 GeV [19]), we have from (5.6) the following upper bound:

$$g_s \frac{4\lambda^2 \rho_d}{(4\pi)^2} < (125 \text{ GeV})^2, \quad (6.1)$$

where the defect density ρ_d and the coupling constant λ are defined by (2.1) and (3.1a), respectively.

In Sec. V B, we already mentioned the possibility that a single energy scale E_{foam} controls the small-scale structure of spacetime and, therefore, sets the parameters of our model (3.1a),

$$\lambda \sim 1/E_{\text{foam}}, \quad (6.2a)$$

$$\rho_d \sim (E_{\text{foam}})^4. \quad (6.2b)$$

With the scenario (6.2) and $g_s \sim 10^{-2}$, bound (6.1) gives

$$E_{\text{foam}} < (10^{-2}/g_s)^{1/2} 8 \text{ TeV}. \quad (6.3)$$

In this case, the picture is that the spacetime defects have an effective size [of order $1/m_\sigma \sim \lambda^{-1} (\rho_d)^{-1/2}$] and typical distance between neighbouring defects [of order $(\rho_d)^{-1/4}$] with the same order of magnitude, $1/E_{\text{foam}}$. This single length scale is, however, very much larger than the Planck length $1/E_P$, with $E_P \equiv G^{-1/2} \approx 1.22 \times 10^{16} \text{ TeV}$.

Scenarios different from (6.2) are also possible. For example, one scenario has $\lambda \sim 1/E_P$ and $\rho_d < (10^{-2}/g_s) (8 \text{ TeV})^2 (E_P)^2$. Ultimately, only the derivation of our model (3.1) from the underlying spacetime (assuming the model to be relevant at all) can decide between the different scenarios.

From a general perspective, it may be of interest to have found another possible origin of mass, barring questions of naturalness and the unknown nature of quantum spacetime. The toy model considered here is rather simple in that it only gives mass to scalars and Dirac fermions. More difficult would be spacetime-defect mass generation for the Weyl fermions and the gauge bosons of a chiral gauge theory (such as the Standard Model). For the Weyl fermions, we may consider replacing the single real scalar field $\sigma(x)$ of our model by a complex scalar field $\Sigma(x)$ in an appropriate representation of the gauge group and using this Σ in gauge-invariant Yukawa terms. For the gauge bosons, perhaps the spacetime-defect mechanism can be merged with a modified version of the Higgs mechanism.

Appendix A: Random-phase approximation

Let us assume that we have to evaluate an integral of the form

$$\int d^4p d^4q f(p, k) j(p) j^*(q), \quad (\text{A1})$$

where $j(p)$ is defined by

$$j(p) \equiv \sum_n \epsilon_n e^{ipx_n}, \quad (\text{A2})$$

with random x_n and ϵ_n as discussed in Secs. II and III, respectively. For the product at different momenta, we have

$$j(p)j^*(q) = \sum_{m,n} \epsilon_m \epsilon_n e^{ipx_n - iqx_m}. \quad (\text{A3})$$

This product can be split in two terms,

$$j(p)j^*(p) = \sum_n e^{ix_n(p-q)} + \sum_{m \neq n} \epsilon_m \epsilon_n e^{ipx_n - iqx_m}. \quad (\text{A4})$$

As discussed in Sec. IIIB, the charges of individual defects are chosen in a such way that the total charge is zero. Let us take two defect cores at the points x_n and x_m , each of charge +1. We consider now near neighbors at the points $x_n + \delta x_n$ and $x_m + \delta x_m$ such that each of them has charge -1. Since the effective volume occupied by one defect is of the order $1/\rho_d$, the characteristic distance between neighboring defects is of order $\rho_d^{-1/4} \equiv l_d$. The second term in (A4) can then be Taylor expanded in terms of l_d ,

$$\begin{aligned} \sum_{m \neq n} \epsilon_m \epsilon_n e^{ipx_n - iqx_m} &= \frac{1}{4} \sum_{m \neq n} \left(e^{ipx_n} e^{-iqx_m} + e^{ip(x_n + \delta x_n)} e^{-iq(x_m + \delta x_m)} \right. \\ &\quad \left. - e^{ip(x_n + \delta x_n)} e^{-iqx_m} - e^{ipx_n} e^{-iq(x_m + \delta x_m)} \right) = \mathcal{O}[p \delta x_n q \delta x_m] = \mathcal{O}[|p| |q| l_d^2]. \end{aligned} \quad (\text{A5})$$

Since the distribution of defects is assumed to be dense, this cross sum is suppressed by the small distance l_d [9, 11]. Therefore, we can approximate the product (A4) by

$$j(p)j^*(p) \approx \sum_n e^{ix_n(p-q)}. \quad (\text{A6})$$

If we take into account that the points x_n are randomly distributed by a Poisson sprinkling process, we expect only significant contributions to (A6) if $p \approx q$. This suggests to replace the expression (A6) by a delta function, which becomes exact under the assumption (4.4). We, finally, have

$$j(p)j^*(p) \approx \rho_d \int d^4z e^{iz(p-q)} = (2\pi)^4 \rho_d \delta^{(4)}(p-q). \quad (\text{A7})$$

-
- [1] J.A. Wheeler, “Geons,” *Phys. Rev.* **97**, 511 (1955).
 - [2] J.A. Wheeler, “On the nature of quantum geometrodynamics,” *Annals Phys.* **2**, 604 (1957).
 - [3] S.W. Hawking, “Space-time foam,” *Nucl. Phys. B* **144**, 349 (1978).
 - [4] S.W. Hawking, D.N. Page, and C.N. Pope, “The propagation of particles in space-time foam,” *Phys. Lett. B* **86**, 175 (1979).
 - [5] S.W. Hawking, D.N. Page, and C.N. Pope, “Quantum gravitational bubbles,” *Nucl. Phys. B* **170**, 283 (1980).
 - [6] S. Bernadotte and F.R. Klinkhamer, “Bounds on length-scales of classical spacetime foam models,” *Phys. Rev. D* **75**, 024028 (2007); arXiv:hep-ph/0610216.
 - [7] F.R. Klinkhamer and M. Schreck, “New two-sided bound on the isotropic Lorentz-violating parameter of modified Maxwell theory,” *Phys. Rev. D* **78**, 085026 (2008); arXiv:0809.3217.
 - [8] J. Collins, A. Perez, D. Sudarsky, L. Urrutia, and H. Vucetich, “Lorentz invariance and quantum gravity: an additional fine-tuning problem?,” *Phys. Rev. Lett.* **93**, 191301 (2004); arXiv:gr-qc/0403053.
 - [9] F.R. Klinkhamer and C. Rupp, “Space-time foam, CPT anomaly, and photon propagation,” *Phys. Rev. D* **70**, 045020 (2004); arXiv:hep-th/0312032.
 - [10] F.R. Klinkhamer and C. Rupp, “Photon-propagation model with random background field: Length scales and Cherenkov limits,” *Phys. Rev. D* **72**, 017901 (2005); arXiv:hep-ph/0506071.
 - [11] M. Schreck, F. Sorba, and S. Thambyahpillai, “Simple model of pointlike spacetime defects and implications for photon propagation,” *Phys. Rev. D* **88**, 125011 (2013); arXiv:1211.0084.
 - [12] S. Hossenfelder, “Phenomenology of spacetime imperfection I: nonlocal defects,” *Phys. Rev. D* **88**, 124030 (2013); arXiv:1309.0311.
 - [13] S. Hossenfelder, “Phenomenology of spacetime imperfection II: local defects,” *Phys. Rev. D* **88**, 124031 (2013);
 - [14] F. Dowker, J. Henson, and R.D. Sorkin, “Quantum gravity phenomenology, Lorentz invariance and discreteness,” *Mod. Phys. Lett. A* **19**, 1829 (2004); arXiv:gr-qc/0311055.
 - [15] L. Bombelli, J. Henson, and R.D. Sorkin, “Discreteness without symmetry breaking: a theorem,” *Mod. Phys. Lett. A* **24**, 2579 (2009); arXiv:gr-qc/0605006.
 - [16] J. Henson, “The causal set approach to quantum gravity,” in “Oriti, D. (ed.): *Approaches to Quantum Gravity* (Cambridge University Press, 2006), pp. 393-413; arXiv:gr-qc/0601121.
 - [17] R.P. Feynman, “Space-time approach to quantum electrodynamics,” *Phys. Rev.* **76**, 769 (1949).
 - [18] L.H. Ford and T. Yoshimura, “Mass generation by selfinteraction in non-minkowskian space-times,” *Phys. Lett. A* **70**, 89 (1979).
 - [19] C. Patrignani *et al.* (Particle Data Group), “The review of particle physics (2016),” *Chin. Phys. C* **40**, 100001 (2016) [available from <http://pdg.lbl.gov/>].